

Classes of Some Hypersurfaces in the Grothendieck Ring of Varieties

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1. Introduction

Let k be a field, let Var_k denote the category of varieties over k . The Grothendieck ring of varieties, denoted by $K_0(Var_k)$, is a ring whose first appearance in the literature dates back to 1964, in a letter of A. Grothendieck to J. P. Serre [15]. Even though this ring has been studied from many aspects since then, still only little is known about it. In the article [26], B. Poonen proves that if the base field k is of characteristic zero, then $K_0(Var_k)$ is not a domain, by providing some example of zero divisors. Another example for the zero divisors of $K_0(Var_k)$ was given by J. Kollár in [19]. F. Heinloth [6] defined $K_0(Var_k)$ in a different way, introducing blow up relations on smooth projective varieties, and proved that this definition is equivalent to the classical one. M. Larsen and V. A. Lunts [20] established an interesting relation of $K_0(Var_k)$ with stable rationality, which is a notion weaker than rationality.

Let X be a hypersurface of degree d in \mathbb{P}_k^n , where $d \leq n$. Let $[X]$ denote its class in $K_0(Var_k)$. The main question that concerns us is whether it is true that $X(k)$ is nonempty if and only if

$$[X] \equiv 1 \quad \text{modulo} \quad [\mathbb{A}_k^1] \quad (1.0.1)$$

in $K_0(Var_k)$, or not. We explain and prove that for some certain types of hypersurfaces this question has a positive answer.

Now let us give some more details about the next sections. In Section 2 are presented the necessary background material and in particular the known results about $K_0(Var_k)$ which will be used in the following sections. We give the definition of $K_0(Var_k)$ with the usual scissor relations

$$[X] = [X \setminus Y] + [Y],$$

if $Y \subset X$ is closed. Then we introduce some of the basic properties of this ring which will be helpful for our calculations. We also include the alternative definition by F. Heinloth [6, Theorem 3.1] in the case that $char(k) = 0$, that describes $K_0(Var_k)$ as the free group generated by smooth projective varieties modulo the blow up relations

$$[Bl_Y X] - [\pi^{-1}(Y)] = [X] - [Y]$$

whenever $Y \subset X$ is a smooth closed center of a blow up $\pi : Bl_Y X \rightarrow X$. Then we briefly give the idea of her proof to explain why these two definitions of $K_0(Var_k)$ are equivalent. In Subsection 2.1 we define stable birational equivalence of irreducible varieties, and we briefly recall a result of M. Larsen and V. A. Lunts [20, Proposition 2.7], that gives the isomorphism

$$K_0(Var_k)/I \cong \mathbb{Z}[SB]$$

where $char(k) = 0$, I is the principal ideal generated by the class of the affine line \mathbb{A}_k^1 and $\mathbb{Z}[SB]$ is the free group generated by the equivalence classes of stably birational smooth complete irreducible varieties. This is a very important result as it allows to examine $K_0(Var_k)$ without having to deal with relations. We also include some basic examples, to make it clear what are the implications of this result and what is not implied, and also is not true in general.

In Section 3 we introduce Question 3.7, which is our main problem, and the motivational background that makes it worthy of interest. We recall the definition of the category of motives, denoted by \mathcal{M}_k , where by motives we mean effective Chow motives with \mathbb{Q} coefficients, if not otherwise stated. We show that one gets a decomposition

$$h(X) = \mathbb{Q}(0) \oplus N \otimes \mathbb{Q}(-1) \tag{1.0.2}$$

of the motive associated to the hypersurface X of degree $d \leq \dim X + 1$ in \mathcal{M}_k , where N is some motive in \mathcal{M}_k . In order to see this, we use and thus state a proposition that is proven by A. Chatzistamatiou in [10, Proposition 1.2]. This proposition demonstrates that if the zero-dimensional Chow group $CH_0(\tilde{h}(X))$ of the reduced motive $\tilde{h}(X)$ vanishes, then we have the decomposition 1.0.2 in \mathcal{M}_k . Thanks to a theorem proved by A. A. Roitman in [27, Theorem 2], which we also state in this section, we in fact have that this Chow group is equal to zero. However one does not have in general the decomposition 1.0.2 when instead the integral Chow motives are considered. In fact if there were a decomposition similar to 1.0.2 over \mathbb{Z} , i.e. if one had $\tilde{h}(X) \cong N \otimes \mathbb{Z}(-1)$ for some Chow motive N with

integral coefficients, it would imply the equivalence 1.0.1 in $K_0(\text{Var}_k)$. To see this, we explain the connection between $K_0(\text{Var}_k)$ and $K_0(\mathcal{M}_k)$, the Grothendieck ring of motives, via the ring homomorphism

$$\chi : K_0(\text{Var}_k) \rightarrow K_0(\mathcal{M}_k)$$

given by $\chi([X]) = [h(X)]$ for X smooth projective. Thus we see the correspondence between Question 3.7 and the decomposition that exists in \mathcal{M}_k . This completes the introduction of the general setting of the problem. In the remaining sections we study particular instances in which the Question 3.7 receive a positive answer.

The main purpose of Section 4 is to figure out what happens in the case $d = 2$. We recall the necessary known results about the rationality of a quadric, and we include a lemma from linear algebra about nondegenerate quadratic forms with a nontrivial solution, of which we will make use in this section and in Section 7. Finally we state as a theorem that in the case $d = 2$ the answer for Question 3.7 is positive, and then prove it.

In Section 5 we study the case of hypersurfaces that are either a union of d hyperplanes, or become a union of hyperplanes after a finite Galois base change. We state and prove that these hypersurfaces provide an affirmative example to Question 3.7, in fact we show that they always have k -rational points, and their classes are always equivalent to 1 modulo the class of affine line. Note that for one hyperplane the answer for our main question is immediately seen to be positive: A hyperplane is isomorphic to the projective space, and it always has k -rational points.

The next natural case to consider would be cubic hypersurfaces, i.e. the case where $d = 3$. For a smooth cubic hypersurface of dimension ≥ 3 , it is still unknown in general if X is (stably) rational or not. C. H. Clemens and P. A. Griffiths [12] proved that a smooth cubic threefold, i.e. a smooth cubic in \mathbb{P}_k^4 is not rational but only unirational, however the question of stable rationality is open also in this case. On the other hand the case of singular cubic hypersurfaces is somewhat easier to deal with. In Section 6 we mainly consider a cubic hypersurface with a k -rational singular point, and we prove that its class in $K_0(\text{Var}_k)$ has the expected form, according to Question 3.7.

In the case of quartic hypersurfaces, our main question is involved only of those ones of dimension 4 or higher. In Section 7 we examine an example of singular quartics: union of two quadrics, one of which is smooth. Note that in this section we assume k to be algebraically closed. Hence our example has always a k -rational point. We compute the class of the intersection of the two quadrics which consist the quartic, and prove that it is equivalent to the class of a singular cubic hypersurface of one smaller dimension, modulo the class of affine line. Then we proceed by using the results of previous section and conclude that the given quartic hypersurface has its class in $K_0(\text{Var}_k)$ in the form which was conjectured to be in Question 3.7.

After the completion of this thesis, we came to know of an independent work submitted on Arxiv by X. Liao [21], which has a significant intersection with what we have done. Whereas our main question is based on the relation between the existence of a k -rational point and the class in $K_0(\text{Var}_k)$, with k an arbitrary field, in [21] k is assumed to be algebraically closed of characteristic zero and the focus is on the " \mathbb{L} -rationality". Other than this, the main parts of this work that differ from [21] are the following theorems where we prove that Question 3.7 is positively answered: Theorem 5.3, which is about a hypersurface X over a field k such that $X \times_k L$ is a union of $d \leq n$ hyperplanes where L/k is a finite Galois extension, and Theorem 7.1, where we study the union of two quadrics, one of which is assumed to be smooth, over an algebraically closed field k of characteristic zero. It is also worth mentioning that after this thesis was submitted, Nguyen L. D. T. gave in [24] a counter-example to the Conjecture 26 in [9], which is also a negative answer to Question 3.7. However, we still do not know whether over an algebraically closed field of characteristic zero, a hypersurface of degree $d \leq n$ in \mathbb{P}^n is congruent to 1 mod \mathbb{L} in the Grothendieck ring over the base field.

2. Grothendieck Ring of Varieties

Let us recall some definitions and properties that will be used.

DEFINITION 2.1. [25, Definition 2.1] *Let k be a field. Let Var_k denote the category of varieties over k . Note that what we mean here by a variety over k is a reduced separated scheme of finite type over k . The Grothendieck group $K_0(\text{Var}_k)$ is the abelian group generated by isomorphism classes of varieties over k , with the relation*

$$[X] = [X \setminus Y] + [Y],$$

if $Y \subset X$ is a closed subvariety. The ring structure is defined by the product

$$[X] \cdot [Z] := [(X \times_k Z)_{\text{red}}]$$

where $(X \times_k Z)_{\text{red}}$ denotes the reduced scheme associated to $X \times_k Z$.

Note that the zero of this ring is $0 := [\emptyset]$, and the multiplicative identity is $1 := [\text{Spec}(k)]$. We denote by \mathbb{L} the class of \mathbb{A}_k^1 in $K_0(\text{Var}_k)$. Hence for any $n \geq 0$ one has $[\mathbb{A}_k^n] = \mathbb{L}^n$, and also $[\mathbb{P}_k^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$.

REMARK 2.2. For any $X \in \text{Var}_k$ with connected components $\{X_i\}_{i=1}^n$ we have $[X] = [X_1 \sqcup \cdots \sqcup X_n] = [X_1] + \cdots + [X_n]$ in $K_0(\text{Var}_k)$. Hence the classes of connected varieties over k generate $K_0(\text{Var}_k)$.

REMARK 2.3. We could actually drop the requirement of reducedness in Definition 2.1. Let X be a separated scheme of finite type over k . Applying the scissor relations given in Definition 2.1 to the natural closed immersion $X_{\text{red}} \rightarrow X$ we obtain

$$[X] = [X \setminus X_{\text{red}}] + [X_{\text{red}}] = [X_{\text{red}}].$$

Let $\pi : X \rightarrow Y$ be a morphism of varieties over k . Recall that π is said to be a Zariski locally trivial fibration with fiber F if each closed point $y \in Y$ has a Zariski open neighborhood U such that the pre-image $\pi^{-1}(U)$ is isomorphic over U to $F \times_k U$. We will frequently use the following remark for our calculations in $K_0(\text{Var}_k)$.

REMARK 2.4. A Zariski locally trivial fibration $\pi : X \rightarrow Y$ becomes trivial in the Grothendieck ring of varieties, i.e. one gets $[X] = [F] \cdot [Y]$. This follows from the defining relations of $K_0(\text{Var}_k)$, using induction on the number of open neighborhoods U_i in Y which has $\pi^{-1}(U_i) = F \times_k U_i$, that covers Y .

Consider for instance the blow up of a smooth projective variety X along a closed smooth subvariety $Z \subset X$ of codimension r . Then the exceptional divisor E is a projective bundle of rank $r - 1$ over Z , i.e. the restriction of the blow up map to E is a Zariski locally trivial fibration with fibers \mathbb{P}_k^{r-1} . Hence by Remark 2.4, we get

$$[E] = [\mathbb{P}_k^{r-1}] \cdot [Z].$$

Note that for such a blow up we also have

$$[Bl_Z X] - [E] = [X] - [Z]$$

in $K_0(Var_k)$. This equation actually yields another definition for the Grothendieck ring of varieties, which is proven to be an equivalent presentation of $K_0(Var_k)$, by F. Heinloth in her paper [6].

REMARK 2.5. ([6, Theorem 3.1]) Let k be a field of characteristic zero, and let $K_0^{bl}(Var_k)$ denote the abelian group generated by the isomorphism classes of smooth projective varieties with the relations $[\emptyset]_{bl} = 0$ and $[Bl_Z X]_{bl} - [E]_{bl} = [X]_{bl} - [Z]_{bl}$ whenever $Bl_Z X$ is the blow up of a smooth projective variety X along a smooth closed subvariety Z with the exceptional divisor E . Then F. Heinloth proves that the ring homomorphism

$$\begin{aligned} K_0^{bl}(Var_k) &\longrightarrow K_0(Var_k) \\ [X]_{bl} &\longmapsto [X] \end{aligned}$$

is an isomorphism.

Idea of proof. First of all it is shown that $K_0(Var_k)$ is generated by smooth varieties with the scissors relations $[X] = [X \setminus Y] - [Y]$ if Y is a smooth closed subvariety of X , this is proven by F. Heinloth by stratification with smooth varieties. Since $Bl_Z X \setminus E \cong X \setminus Z$ for every blow up $Bl_Z X$ of a variety X along a smooth center, the ring homomorphism of Remark 2.5 is well defined. To prove that it is an isomorphism, F. Heinloth constructs an inverse using Hironaka's theorem to complete a smooth variety X with a simple normal crossing divisor D , then define an inverse which maps X to the alternating sum of the classes in $K_0^{bl}(Var_k)$ of l -fold intersections of the irreducible components of D , for $0 \leq l \leq n$ with 0-fold intersection taken to be the completion itself. To prove this sum is independent of the choice of completion, the weak factorization theorem ([2, Theorem 0.1.1], is used. \square

2.1. Stable Rationality.

DEFINITION 2.6. Let X, Y be irreducible varieties over a field k . Then X and Y are called stably birational if there exist integers $m, n \geq 0$ such that $X \times_k \mathbb{P}_k^n$ is birational to $Y \times_k \mathbb{P}_k^m$ over k . Furthermore, an irreducible variety X is said to be stably rational over k , if it is stably birational to $\mathbb{P}_k^{dim X}$.

The question about the existence of nonrational stably rational varieties has been answered negatively in the article [5]: First example is an affine conic bundle X over an algebraically nonclosed field k with $char(k) \neq 2$, that is given by the equation $y^2 - az^2 = P(x) \neq 0$ where $P(x) \in k[x]$ is a cubic irreducible separable polynomial whose discriminant is $\Delta(P) = a \in k^*$ and a is not a square in k . It is proven in [5] that X is nonrational but $X \times \mathbb{A}_k^3$ is birational to \mathbb{A}_k^5 . Following this example they also construct a nonrational stably rational threefold over an algebraically closed field.

Note that stable birationality defines an equivalence relation \sim_{SB} on the set of smooth complete irreducible varieties over k . Let SB denote the

set of equivalence classes of this relation. We will denote by $\mathbb{Z}[SB]$ the free abelian group generated by SB .

In the article of M. Larsen and V. A. Lunts [20], for a smooth complete irreducible variety it is proven that stable rationality is a necessary and sufficient condition for being equivalent to 1 modulo the class of affine line in $K_0(Var_{\mathbb{C}})$. Let us recall this important result and the general idea of its proof given by Larsen and Lunts.

THEOREM 2.7. [20, Theorem 2.3] *Let G be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. Denote by \mathcal{M} the multiplicative monoid of isomorphism classes of smooth complete irreducible varieties over \mathbb{C} . Let*

$$\Psi : \mathcal{M} \longrightarrow G$$

be a homomorphism of monoids such that

- (1) $\Psi([X]) = \Psi([Y])$ if X and Y are birational;
- (2) $\Psi(\mathbb{P}_{\mathbb{C}}^n) = 1$ for all $n \geq 0$.

Then there exists a unique ring homomorphism

$$\Phi : K_0(Var_{\mathbb{C}}) \longrightarrow \mathbb{Z}[G]$$

such that $\Phi([X]) = \Psi([X])$ for $[X] \in \mathcal{M}$.

REMARK 2.8. ([20, Subsection 2.1],[6, Theorem 3.1- (sm)])

- (1) Every variety can be stratified by a finite number of smooth varieties. Indeed we can write any variety X as the union of its singular locus $Sing(X)$ and the complement $X \setminus Sing(X)$. Consider $Sing(X)$ with the reduced induced scheme structure. If $Sing(X)$ is not a smooth variety, we repeat what we have just done, and we will be done in finitely many steps because of finite dimension. Since $Sing(X)$ is a closed subvariety of X , this implies that $[X]$ can be written as a finite sum of the classes of smooth varieties. Hence the classes of smooth varieties, considering the fact that connected implies irreducible, the classes of smooth irreducible varieties generate $K_0(Var_k)$.
- (2) Assume that $char(k) = 0$. Then $K_0(Var_k)$ is generated by the classes of projective smooth irreducible varieties over k : let X be an irreducible variety of dimension n , over k . We will use induction on the dimension of X . Let $U \subset X$ be an affine open subvariety. Then the complement of U is a union of closed subvarieties of strictly smaller dimension. Thus we may assume that X itself is affine. Now let us consider a projective completion \overline{X} of X . By Hironaka's theorem (cf. [1, Theorem 0.1.]) there exists a proper birational morphism $f : X_1 \rightarrow \overline{X}$ over k such that X_1 is a projective smooth variety over k . Let $D \subset \overline{X}$ be the discriminant locus of f , i.e let $X_1 \setminus f^{-1}(D)$ be the maximal open subset on which f is an isomorphism. Then we get that

$$[\overline{X}] = [X_1] - [f^{-1}(D)] + [D]$$

and we can conclude by using the induction hypothesis since we have

$$[X] = [X_1] - [f^{-1}(D)] + [D] - [\overline{X} \setminus X].$$

Idea of the proof of Theorem 2.7. Larsen and Lunts construct the ring homomorphism Φ by induction on the dimension of varieties, with some assertions to prove the existence and unambiguity of the construction. They start with putting $\Phi([X]) := \Psi([X])$, for X an irreducible smooth complete variety. Note that this already ensures the uniqueness of Φ , since $K_0(\text{Var}_{\mathbb{C}})$ is generated by the classes of irreducible smooth complete varieties over \mathbb{C} . Then by using Hironaka's theorem, and the induction hypotheses they define Φ for any smooth variety X with connected components X_1, \dots, X_k as below

$$\Phi([X]) := \sum \Phi(\overline{X}_i) - \sum \Phi(\overline{X}_i \setminus X_i)$$

where $X_i \hookrightarrow \overline{X}_i$ is an open embedding of X_i in a smooth complete irreducible variety \overline{X}_i , for each i . Note that such embeddings exist by Hironaka's theorem. To prove that this definition is independent of the choice of \overline{X}_i 's, they use the weak factorization theorem of Włodarczyk [2, Theorem 0.1.1.], together with the fact that a projectivization $\overline{E} \rightarrow X$ of a vector bundle $E \rightarrow X$ is Zariski locally trivial. Finally, using the induction hypotheses they define

$$\Phi([X]) := \Phi([X \setminus \text{Sing}(X)]) + \Phi([\text{Sing}(X)]),$$

for any arbitrary variety X . They conclude by proving the assertions. \square

Now taking $G = SB$ in Theorem 2.7 one gets the ring homomorphism

$$\Phi_{SB} : K_0(\text{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}[SB],$$

which is clearly surjective.

PROPOSITION 2.9. [20, Proposition 2.7] *With the assumptions and notations of Theorem 2.7, the kernel of the ring homomorphism Φ_{SB} defined above is the principal ideal generated by the class \mathbb{L} .*

Idea of the proof. Let $\text{Ker}(\Phi_{SB})$ denote the kernel of Φ_{SB} . Then $\mathbb{L} \in \text{Ker}(\Phi_{SB})$ since $\Phi_{SB}([\mathbb{P}_k^1]) = \Phi_{SB}(1 + \mathbb{L}) = 1$. Hence $K_0(\text{Var}_{\mathbb{C}}) \cdot \mathbb{L} \subset \text{Ker}(\Phi_{SB})$. Let $a \in \text{Ker}(\Phi_{SB})$. By Remark 2.8,(2), one can write a as a linear combination of the classes of finitely many smooth complete varieties. Using this, Larsen and Lunts reduce to the case $a = [X] - [Y]$, where X, Y are smooth, complete and stably birational. Since $[X \times_k \mathbb{P}_{\mathbb{C}}^n] - [X] \in K_0(\text{Var}_{\mathbb{C}}) \cdot \mathbb{L}$, one may assume without loss of generality that X, Y are birational. By weak factorization theorem [2, Theorem 0.1.1], one can make a further reduction to the case where X is a blow up of Y along a smooth center. Then it follows that $a = [X] - [Y] \in K_0(\text{Var}_{\mathbb{C}}) \cdot \mathbb{L}$. \square

REMARK 2.10. For X connected smooth projective, Proposition 2.9 implies that X is stably rational if and only if $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_{\mathbb{C}})$. Note that however we have no longer this implication for X singular or non-complete in general. This is because Φ_{SB} , by construction, sends only the

class of a smooth complete variety to its own equivalence class in $\mathbb{Z}[SB]$, i.e. for X not smooth complete, the equivalence $X \sim_{SB} \text{Spec}(\mathbb{C})$ does not necessarily imply that $\Phi_{SB}([X]) = 1$.

As it is pointed out by J. Kollár in [19], the result of Larsen and Lunts holds in fact over any field of characteristic zero: this follows from the presentation of $K_0(\text{Var}_k)$ by means of the blow up relations (see Remark 2.5).

The following proposition, which follows directly from Theorem 2.7, demonstrates a noteworthy connection between $K_0(\text{Var}_k)$ and k -rational points.

PROPOSITION 2.11. *Let k be a field of characteristic zero, let X be a smooth connected complete variety of dimension m over k . If $[X] \equiv 1 \pmod{\mathbb{L}}$, then Proposition 2.9 implies that $X(k) \neq \emptyset$.*

PROOF. As it is mentioned in Remark 2.10, we know that X is stably rational, i.e. there is an $a \geq 0$ such that $X \times \mathbb{P}_k^a$ is rational. Let $\phi : \mathbb{P}_k^{m+a} \dashrightarrow X \times \mathbb{P}_k^a$ be a birational map, and let $U \subset \mathbb{P}_k^{m+a}$ be the open subset on which the map ϕ is defined, and is an isomorphism. It is then guaranteed that $U(k) \neq \emptyset$ since we assume $\text{char}(k) = 0$, which implies that k is an infinite field. Hence $X \times \mathbb{P}_k^a$ has a k -rational point, which implies that X has a k -rational point. \square

As an effort to make the point of Remark 2.10 clearer, let us illustrate some negative examples. For non-complete (stably) rational varieties, consider simply the affine space \mathbb{A}_k^n . Although it is rational, one has the equivalence $[\mathbb{A}_k^n] \equiv 0 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$, since $[\mathbb{A}_k^n] = \mathbb{L}^n$, for all $n \geq 0$.

Below we give an easy example of singular varieties whose equivalence classes are $1 \pmod{\mathbb{L}}$ even though they are not stably rational.

EXAMPLE 2.12. *Let $X \subset \mathbb{P}_k^n$ be a nonrational cone with vertex x over a hypersurface $Z \subset \mathbb{P}_k^{n-1}$. Note that a cone is always singular, and its vertex is a k -rational point. Therefore after a change of coordinates, we can assume that $x = [0 : \cdots : 0 : 1] \in X$. The projection*

$$\begin{aligned} \mathbb{P}_k^n \setminus \{x\} &\longrightarrow \mathbb{P}_k^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_{n-1}] \end{aligned}$$

is a Zariski locally trivial affine fibration. The following is a cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus \{x\} & \longrightarrow & \mathbb{P}_k^{n-1} \\ \uparrow & & \uparrow \\ X \setminus \{x\} & \longrightarrow & Z \end{array}$$

Thus we get that $X \setminus \{x\} \rightarrow Z$ is also a Zariski locally trivial affine fibration, which implies by Remark 2.4 that $[X \setminus \{x\}] = \mathbb{L} \cdot [Z]$. Hence

$$[X] = 1 + \mathbb{L} \cdot [Z],$$

and we see that $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$.

Now let us consider the case of a stably rational variety that is not smooth. In the below examples one sees that such a variety need not have its class in $K_0(\text{Var}_k)$ equivalent to 1 modulo \mathbb{L} .

EXAMPLE 2.13. Let $X \subset \mathbb{P}_{\mathbb{C}}^2$ be the projective nodal curve, more precisely the projective hypersurface given by the homogenization of the equation

$$y^2 = x^3 + x^2.$$

It is well known that X is a rational curve with a unique singular point at the origin O . Let $\pi : \tilde{X} \rightarrow X$ be the blow up of X at O , and let $E := \pi^{-1}(O)$. Since the exceptional divisor E consists of two disjoint \mathbb{C} -rational points, we have $[E] = 2$ in $K_0(\text{Var}_{\mathbb{C}})$. As the blow up \tilde{X} is the normalization of X , it is a projective smooth rational irreducible curve, and therefore it is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$, and we get $[\tilde{X}] = 1 + \mathbb{L}$. Thus we obtain

$$\begin{aligned} [X] &= [X \setminus O] + [O] \\ &= [\tilde{X} \setminus E] + [O] \\ &= [\tilde{X}] - [E] + [O] \\ &\equiv 0 \text{ mod } \mathbb{L}. \end{aligned}$$

The following example illustrates that even a normal singular variety, i.e. a variety with singularities at least codimension 2, might have its class in $K_0(\text{Var}_k)$ not equivalent to 1 modulo \mathbb{L} .

EXAMPLE 2.14. Let X be a rational normal projective surface over an algebraically closed field k with a unique singular point $x \in X$, and let $\tilde{X} \rightarrow X$ be the minimal resolution of singularities of X where \tilde{X} is the blow up of X at x . Let moreover the exceptional divisor $E := \pi^{-1}(x)$ be a union of three lines in \mathbb{P}_k^2 which form a cycle, i.e. the intersection of all three lines is empty. Note that one has indeed such a surface, and below we will sketch its construction. But let us first show that $[X] \not\equiv 1 \text{ mod } \mathbb{L}$ in $K_0(\text{Var}_k)$. Being the blow up of a projective rational variety, \tilde{X} is also rational and projective, since it is smooth we can conclude by Remark 2.10 that $[\tilde{X}] \equiv 1 \text{ mod } \mathbb{L}$ in $K_0(\text{Var}_k)$. Also note that

$$[E] = 3[\mathbb{P}_k^1] - 3 = 3\mathbb{L} \equiv 0 \text{ mod } \mathbb{L}$$

in $K_0(\text{Var}_k)$. Thus we get

$$\begin{aligned} [X] &= [\tilde{X} \setminus E] + [\{x\}] \\ &= [\tilde{X}] - [E] + 1 \\ &\equiv 2 - [E] \text{ mod } \mathbb{L} \\ &\equiv 2 \text{ mod } \mathbb{L} \end{aligned}$$

in $K_0(\text{Var}_k)$.

Let us now roughly describe the construction of X of Example 2.14. Let $L_1, L_2, L_3 \subset \mathbb{P}_k^2$ be distinct lines such that the intersection $L_1 \cap L_2 \cap L_3$ is empty, so that they form a cycle. Now we mark four distinct points

$\{p_{i1}, p_{i2}, p_{i3}, p_{i4}\} =: P_i$ on each L_i , where none of the p_{ij} 's is included in $L_{i'}$ with $i \neq i'$. Write

$$S := P_1 \cup P_2 \cup P_3.$$

Let $\tilde{X} := \text{Bl}_S \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^2$ be the blow up of \mathbb{P}_k^2 at S . We claim that

$$R := L_1 \cup L_2 \cup L_3 \subset \tilde{X}$$

is contractible, i.e. there exists a birational morphism

$$f : \tilde{X} \longrightarrow X$$

such that $f(R) = \{x\}$, where x is a point of X , and $f : \tilde{X} \setminus R \rightarrow X \setminus \{x\}$ is an isomorphism. Recall that to be contractible a curve needs to be negative definite which means that the matrix of the irreducible components of the curve is negative definite: in our example the matrix is

$$(L_i \cdot L_j)_{i,j} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}$$

since $L_i \cdot L_i = -3$ for all i , and $L_i \cdot L_j = 1$ for all $i \neq j$. Thus R is negative definite.

Denote by E_{ij} the exceptional divisor of p_{ij} in \tilde{X} . Let H be a general line in \mathbb{P}_k^2 . Consider the divisor $D = 4H - \sum_{i=1}^3 \sum_{j=1}^4 E_{ij}$ on \tilde{X} . One can prove

that the complete linear system $|mD|$ is base point free for a sufficiently large integer $m \gg 0$, and therefore $|mD|$ defines a morphism

$$f : \tilde{X} \longrightarrow \mathbb{P}_k^N.$$

Note that we have $D \cdot L_i = 4H \cdot L_i - \sum_{j=1}^4 E_{ij} \cdot L_i = 0$ for all i . Besides

$D^2 = 16 - 12 = 4 > 0$, and $D \cdot E_{ij} = -E_{ij} \cdot E_{ij} = 1$. Let $C \subset \text{Bl}_S \mathbb{P}_k^2$ be any irreducible curve with $C \not\subset \{L_i, E_{ij}\}$ for all i, j . Then we can identify it with the irreducible curve it comes from in the projective plane \mathbb{P}_k^2 . Hence one gets

$$D \cdot C = 4\deg(C) - \sum_{i=1}^3 \sum_{j=1}^4 \text{mult}_{p_{ij}}(C)$$

where $\text{mult}_{p_{ij}}(C)$ denotes the multiplicity of the curve C at the point p_{ij} . By the definition of multiplicity one has $\text{mult}_{p_{ij}}(C \cap L_i) \geq \text{mult}_{p_{ij}}(C)$,

which yields

$$\begin{aligned}
D \cdot C &= 4deg(C) - \sum_{i=1}^3 \sum_{j=1}^4 mult_{p_{ij}}(C) \\
&\geq 4deg(C) - \sum_{i=1}^3 \sum_{j=1}^4 mult_{p_{ij}}(C \cap L_i) \\
&\geq 4deg(C) - 3deg(C) \\
&= deg(C) > 0.
\end{aligned}$$

Thus f contracts precisely R , and the image $X := f(\tilde{X})$ has the desired properties of Example 2.14.

3. Introduction to the Main Question

Let k be a field of characteristic zero, let X be a hypersurface over k of degree d in \mathbb{P}_k^n , where $d \leq n$.

REMARK 3.1. Let us recall that a smooth hypersurface $X \subset \mathbb{P}_k^n$ of degree d is a Fano variety, i.e. its canonical bundle is anti-ample if and only if $d \leq n$.

PROOF. Let ω_X denote the canonical bundle of X , then by adjunction formula [14, Chapter 1, §1] we have

$$\begin{aligned}\omega_X &\cong \omega_{\mathbb{P}_k^n}|_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \\ &= \mathcal{O}_X(d - n - 1)\end{aligned}$$

By assumption we have $d - n - 1 \leq -1$, which implies that the canonical bundle ω_X is anti-ample. \square

3.1. Category of Motives. [3, Chapter 4] Let k be a field, and let \mathcal{V}_k denote the category of smooth projective schemes over k . For $X \in \mathcal{V}_k$, and for a non-negative integer $d \leq \dim X$, recall that the group of codimension d cycles $Z^d(X)$ is the free abelian group generated by the codimension d subvarieties of X . Then the Chow group of X of codimension d is defined as $CH^d(X) := Z^d(X)/\sim$, where \sim denotes the rational equivalence. Note that we consider $CH^d(X)$ to be with \mathbb{Q} -coefficients, and also note that $CH^d(X) := 0$, for all $d < 0$ and $d > \dim X$.

DEFINITION 3.2. Let $X, Y \in \mathcal{V}_k$, and let X_i be the connected components of X . The group of correspondences of degree r from X to Y is defined to be

$$Hom^r(X, Y) := \bigoplus_i CH^{\dim X_i + r}(X_i \times Y).$$

We will simply denote by $Hom(X, Y)$ the group of the correspondences of degree zero.

Let $P \in Hom(X, Y)$, $Q \in Hom(Y, Z)$. Then the composition is given by

$$Q \circ P := (p_{13})_*(p_{12}^*(P) \cdot p_{23}^*(Q))$$

where p_{ij} are the projection maps from $X \times Y \times Z$ to the product of the i -th and j -th factors, and $(p_{ij})_*$, p_{ij}^* are the pull-back and push-forward of p_{ij} in the Chow groups, respectively. Note also that \cdot denotes the intersection product in $CH^{\dim X}(X \times Y \times Z)$. This composition gives $Hom(X, X)$ a \mathbb{Q} -algebra structure.

DEFINITION 3.3. A pair (X, P) with $X \in \mathcal{V}_k$ and $P \in Hom(X, X)$ a projector, i.e. $P = P \circ P$, is called a motive.

Motives form a category denoted by \mathcal{M}_k with morphism groups

$$Hom_{\mathcal{M}_k}((X, P), (Y, Q)) := Q \circ Hom(X, Y) \circ P \subset Hom(X, Y).$$

Note that the identity morphism of a motive (X, P) is the projector P . The sum and the product in \mathcal{M}_k are defined by disjoint union and product:

$$\begin{aligned}(X, P) \oplus (Y, Q) &= (X \sqcup Y, P + Q) \\ (X, P) \otimes (Y, Q) &= (X \times Y, P \times Q).\end{aligned}$$

The motive associated with $X \in \mathcal{V}_k$ is the motive (X, id_X) where $id_X \in Hom(X, X)$ is the class of the diagonal Δ_X in the Chow group. There is a functor

$$h : \mathcal{V}_k^{op} \longrightarrow \mathcal{M}_k$$

given on objects by $h(X) := (X, id_X)$, and on morphisms by $h(\varphi) := [{}^t\Gamma_\varphi]$, the class of the transpose of the graph of $\varphi : Y \rightarrow X$. Note that $\mathbb{Q}(0) := h(Spec(k))$ is the identity for the product and it is called the unit motive. Consider the motive $h(\mathbb{P}_k^1)$. For any $x \in \mathbb{P}_k^1(k)$ we have

$$[\Delta_{\mathbb{P}_k^1}] = [\{x\} \times \mathbb{P}_k^1] + [\mathbb{P}_k^1 \times \{x\}] \quad (3.1.1)$$

in $CH^1(\mathbb{P}_k^1 \times \mathbb{P}_k^1)$. Hence

$$h(\mathbb{P}_k^1) = (\mathbb{P}_k^1, \{x\} \times \mathbb{P}_k^1) \oplus (\mathbb{P}_k^1, \mathbb{P}_k^1 \times \{x\}). \quad (3.1.2)$$

Consider the following morphisms of motives

$$\begin{aligned}\phi_1 &:= [Spec(k) \times \mathbb{P}_k^1] \in Hom_{\mathcal{M}_k}(\mathbb{Q}(0), (\mathbb{P}_k^1, \{x\} \times \mathbb{P}_k^1)), \\ \phi_2 &:= [\{x\} \times Spec(k)] \in Hom_{\mathcal{M}_k}((\mathbb{P}_k^1, \{x\} \times \mathbb{P}_k^1), \mathbb{Q}(0)).\end{aligned}$$

Then we get $\phi_1 \circ \phi_2 = id_{\mathbb{Q}(0)}$ and $\phi_2 \circ \phi_1 = id_{(\mathbb{P}_k^1, \{x\} \times \mathbb{P}_k^1)}$. Thus the first summand of the decomposition 3.1.2 is isomorphic to $\mathbb{Q}(0)$. In a similar way one can show that the latter summand

$$(\mathbb{P}_k^1, \mathbb{P}_k^1 \times \{x\}) =: \mathbb{Q}(-1)$$

is isomorphic to (-1) -twist of $\mathbb{Q}(0)$. By a (-1) -twist, we mean that for all $(X, P) \in \mathcal{M}_k$ one has

$$Hom_{\mathcal{M}_k}((X, P), \mathbb{Q}(-1)) = id_{Spec(k)} \circ Hom^{-1}(X, Spec(k)) \circ P.$$

This motive $\mathbb{Q}(-1)$ is called the Lefschetz motive. Now let

$$\mathbb{Q}(a) := \mathbb{Q}(-1)^{\otimes -a}$$

for $a < 0$, and let $X \in \mathcal{V}_k$ be connected. Then

$$Hom_{\mathcal{M}_k}((X, P) \otimes \mathbb{Q}(a), (Y, Q) \otimes \mathbb{Q}(b)) = Q \circ CH^{dim X - a + b}(X \times Y) \circ P.$$

Similar to $h(\mathbb{P}_k^1)$, one can give a decomposition of $h(X)$ for any $X \in \mathcal{V}_k$. Let $x \in X$ be a closed point with residue field $\kappa(x)$. Consider the degree one cycles

$$\alpha := deg(\kappa(x)/k)^{-1}[\{x\} \times Spec(k)] \in Hom_{\mathcal{M}_k}(h(X), \mathbb{Q}(0))$$

and

$$\beta := [Spec(k) \times X] \in Hom_{\mathcal{M}_k}(\mathbb{Q}(0), h(X)).$$

Then we get

$$\begin{aligned}
\alpha \circ \beta &= \deg(\kappa(x)/k)^{-1} (p_{13})_* (p_{12}^*(\beta) \cdot p_{23}^*(\alpha)) \\
&= \deg(\kappa(x)/k)^{-1} (p_{13})_* ([Spec(k) \times X \times Spec(k)] \cdot \\
&\quad [Spec(k) \times \{x\} \times Spec(k)]) \\
&= \deg(\kappa(x)/k)^{-1} \deg(\kappa(x)/k) [Spec(k) \times Spec(k)] \\
&= id_{\mathbb{Q}(0)}.
\end{aligned}$$

Therefore $id_X = id_{\mathbb{Q}(0)} + (id_X - id_{\mathbb{Q}(0)})$ is an orthogonal decomposition where both summands are projectors. This yields a decomposition

$$h(X) = \mathbb{Q}(0) \oplus \tilde{h}(X) \quad (3.1.3)$$

in \mathcal{M}_k , where $\tilde{h}(X)$ denotes the motive $(X, id_X - id_{\mathbb{Q}(0)})$. Note that this decomposition in general depends on x . However the decomposition $h(\mathbb{P}_k^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ is canonical since the decomposition 3.1.1 of $\Delta_{\mathbb{P}_k^1}$ in $CH^1(\mathbb{P}_k^1 \times \mathbb{P}_k^1)$ is independent of the choice of $x \in \mathbb{P}_k^1(k)$.

Let $K_0(\mathcal{M}_k)$ denote the Grothendieck ring of the category \mathcal{M}_k , this is the group generated by the isomorphism classes of motives with the relation

$$[M \oplus N] - [M] - [N]$$

for all $M, N \in \mathcal{M}_k$. The product is given by

$$[M] \cdot [N] := [M \otimes N].$$

Let $X \in \mathcal{V}_k$, and let Z be a closed smooth subvariety of X . Then one has the following canonical isomorphism of motives

$$h(Bl_Z X) \oplus h(Z) \cong h(X) \oplus h(E)$$

where $Bl_Z X$ is the blow up of X at Z , and E is the exceptional divisor [22, §9]. Hence

$$[h(Bl_Z X)] - [h(E)] = [h(X)] - [h(Z)]$$

in $K_0(\mathcal{M}_k)$. By Remark 2.5, we know also that $K_0(Var_k)$ is generated by smooth projective varieties with the blow up relations. This implies that the functor h induces a ring homomorphism $\chi : K_0(Var_k) \rightarrow K_0(\mathcal{M}_k)$ given by $\chi([X]) = [h(X)]$ for $X \in \mathcal{V}_k$. Note that one has

$$\chi(\mathbb{L}) = [h(\mathbb{P}^1)] - [\mathbb{Q}(0)] = [\mathbb{Q}(-1)].$$

Let us now give the definition for the Chow groups of a motive $M \in \mathcal{M}_k$ by

$$CH^i(M) := Hom_{\mathcal{M}_k}(\mathbb{Q}(-i), M), \text{ and } CH_i(M) := Hom_{\mathcal{M}_k}(M, \mathbb{Q}(-i))$$

for $i \geq 0$ and $CH^i(M) = 0 = CH_i(M)$ for $i < 0$. Note that if $M = h(X)$ for a $X \in \mathcal{V}_k$ then we get simply $CH^i(M) = CH^i(X)$ and $CH_i(M) = CH_i(X)$. Consider a field extension $k \subset L$. There is a functor between \mathcal{M}_k and \mathcal{M}_L given by base change:

$$\begin{aligned}
\times_k L : \mathcal{M}_k &\longrightarrow \mathcal{M}_L \\
(X, P) &\longmapsto (X \times_k L, P \times_k L).
\end{aligned}$$

In his article [10], using the argumentation of [8], Chatzistamatiou proves the following proposition for the motives of which the degree zero Chow groups are zero:

PROPOSITION 3.4. [10, Proposition 1.2] *Let k be a perfect field, and $X \in \mathcal{V}_k$ be connected.*

- (1) *A motive $M = (X, P)$ can be written as $M \cong N \otimes \mathbb{Q}(-1)$ with some motive N if and only if $CH_0(M \times_k L) = 0$ for some field extension L of the function field $k(X)$ of X .*
- (2) *There exists an isomorphism $M \cong N \otimes \mathbb{Q}(a)$ with some motive N and $a < 0$ if and only if $CH_i(M \times_k L) = 0$ for all $i < -a$ and all field extensions $k \subset L$.*

We now explain that the following theorem, which is due to A. Roitman [27], implies for X a hypersurface of degree $d \leq n$ that one gets $CH_0(\tilde{h}(X)) = 0$.

THEOREM 3.5. [27, Theorem 2] *Let k be an algebraically closed field, let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree d with $d \leq n$. Then the subgroup of $CH_0(X)$ of degree zero cycles over \mathbb{Z} , denoted by $CH_0^0(X)$ is zero.*

Hence the degree map

$$\begin{aligned} CH_0(X) \otimes \mathbb{Q} &\longrightarrow \mathbb{Q} \\ \sum_i n_i [Z_i] &\longmapsto \sum_i n_i \end{aligned}$$

is an isomorphism.

Note that the degree map given above is still an isomorphism over an algebraically non-closed field (cf. [7, Lecture 1, Appendix, Lemma 3]). Let k be any field. Recall that

$$CH_0(X_{\bar{k}}) \otimes \mathbb{Q} = \varinjlim CH_0(X_E) \otimes \mathbb{Q},$$

where the direct limit is taken over all finite extensions of k , and where \bar{k} denotes the algebraic closure of k . Let $\pi : X_E \rightarrow X$ be the base change map of a hypersurface X as given in Theorem 3.5, over a finite extension E of k . Then one has the pull-back $\pi^* : CH_0(X) \rightarrow CH_0(X_E)$ and the push-forward homomorphism $\pi_* : CH_0(X_E) \rightarrow CH_0(X)$, and one gets that $\pi_* \circ \pi^*$ is multiplication by $[E : k]$ since E/k is finite. Therefore after tensoring with \mathbb{Q} , $\pi_* \circ \pi^*$, hence π^* becomes injective. By passing to the direct limits, this gives an injection $CH_0(X) \otimes \mathbb{Q} \hookrightarrow CH_0(X_{\bar{k}}) \otimes \mathbb{Q}$. Therefore one gets that the degree map $CH_0(X) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ is an isomorphism, since the degree map over the algebraic closure is an isomorphism by Theorem 3.5.

Now let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree $d \leq n$ over a perfect field k , not necessarily algebraically closed. Then one has

$$\begin{aligned} \mathbb{Q} \cong CH_0(X) &= Hom_{\mathcal{M}_k}(h(X), \mathbb{Q}(0)) \\ &= Hom_{\mathcal{M}_k}(\mathbb{Q}(0) \oplus \tilde{h}(X), \mathbb{Q}(0)) \\ &= \mathbb{Q} \oplus CH_0(\tilde{h}(X)), \end{aligned}$$

hence $CH_0(\tilde{h}(X)) = 0$. Therefore applying Proposition 3.4 to the motive $\tilde{h}(X)$ of Equation 3.1.3, one gets the following decomposition

$$h(X) = \mathbb{Q}(0) \oplus N \otimes \mathbb{Q}(-1) \quad (3.1.4)$$

with some motive $N \in \mathcal{M}_k$.

REMARK 3.6. If one considers the category of motives over \mathbb{Z} , i.e. if the correspondences are Chow groups with integer coefficients, then it is not known if having $CH_0(\tilde{h}(X)) = 0$ implies a decomposition of the form

$$h(X) = \mathbb{Z}(0) \oplus N \otimes \mathbb{Z}(-1) \quad (3.1.5)$$

with some integral motive N .

It is also unknown if one in general gets a corresponding decomposition to 3.1.5, of the class of X in the Grothendieck ring of varieties, note that this is again an integral question. Our main concern will be to search for such decomposition of the classes of hypersurfaces of degree $d \leq n$ in $K_0(Var_k)$. Let us formally ask our main question which is due to H. Esnault. Let $[X]$ denote the equivalence class of X in $K_0(Var_k)$ from now on, unless otherwise stated.

QUESTION 3.7. *For a projective hypersurface $X \subset \mathbb{P}_k^n$ of degree $d \leq n$, does one have $X(k) \neq \emptyset$ if and only if $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(Var_k)$?*

The first example of the varieties that satisfy the assumption of this question would be naturally a hyperplane $H \subset \mathbb{P}_k^n$. Then the answer to the question is clearly yes since H is a linear subspace which is isomorphic to \mathbb{P}_k^{n-1} , and it always has k -rational points, and

$$[H] = [\mathbb{P}_k^{n-1}] = 1 + \mathbb{L} + \cdots + \mathbb{L}_k^{n-1}.$$

Now clearly the spirit of Question 3.7 varies with the degree of X and also with the base field k . Recall that for a field k to be $C1$ means that any homogeneous polynomial $F(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ of degree $d \leq n$ has a nontrivial solution over k . Thus over a $C1$ field Question 3.7 takes the following form

QUESTION 3.8. (cf. [9, Conjecture 26]) *Let k be a $C1$ field, and let X be as in Question 3.7. Is $[X] \equiv 1 \pmod{\mathbb{L}}$?*

In other words, having a positive answer to Question 3.7 over a $C1$ field k implies that for *every* projective hypersurface $X \subset \mathbb{P}_k^n$ of degree $d \leq n$, one has $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(Var_k)$.

3.2. Chevalley-Warning Theorem. Let us consider the case of finite ground fields: we assume for this subsection that $k = \mathbb{F}_q$ with $q = p^f$ for some $f > 0$. For a polynomial $F \in k[x_0, \dots, x_n]$, denote by $N(F)$ the number of its solutions over k . We have the following important well-known theorem about this number $N(F)$.

THEOREM 3.9. (Chevalley-Warning) *Let k be a finite field with $\text{char}(k) = p$, let $F(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ be a homogeneous polynomial of degree*

d such that $0 < d \leq n$. Then

$$1 < N(F) \equiv 0 \pmod{q}.$$

Let us note that the original version of this theorem given by C. Chevalley and E. Warning is modulo p instead of q . However later in [4] it was proven by J. Ax that the equivalence holds also modulo q .

Returning to Question 3.7, let us consider a projective hypersurface $X \subset \mathbb{P}_{\mathbb{F}_q}^n$ of degree $d \leq n$, and let F be the defining polynomial of X . Then we have

$$\sharp(X(\mathbb{F}_q)) = \frac{N(F) - 1}{q - 1}$$

which implies by Theorem 3.9 that

$$\emptyset \neq X(\mathbb{F}_q) \equiv 1 \pmod{q}. \quad (3.2.1)$$

Now let us pass to the ring $K_0(\text{Var}_{\mathbb{F}_q})$: it is known that when the ground field is \mathbb{F}_q , there exists a unique ring homomorphism

$$\sharp : K_0(\text{Var}_{\mathbb{F}_q}) \longrightarrow \mathbb{Z}$$

given by

$$[X] \longmapsto \sharp(X(\mathbb{F}_q)).$$

Via this homomorphism we can rewrite the equivalence 3.2.1 as

$$\sharp([X]) \equiv 1 \pmod{\sharp(\mathbb{L})}. \quad (3.2.2)$$

What we search in Question 3.7 is the existence of this kind of equivalence already in $K_0(\text{Var}_k)$, over any ground field k , therefore our main question can as well be considered as a geometric analog of the Chevalley-Warning theorem.

In the remaining sections we study some particular cases which give a positive answer to Question 3.7, using elementary geometric methods. We show that for the union of d hyperplanes in \mathbb{P}_k^n over any field k with $d \leq n$, and for quadrics over a field of characteristic zero the answer is affirmative. In the case of a nonsingular cubic, the setting is already far more complicated: recall that a variety X of dimension m over a field k is called unirational when there is a rational map $\varphi : \mathbb{P}_k^m \dashrightarrow X$ such that $\varphi(\mathbb{P}_k^m)$ is dense in X and the function field $k(\mathbb{P}_k^m)$ is a separable extension of $k(X)$. It is well known that all smooth cubic hypersurfaces over an algebraically closed field are unirational. More generally, for any cubic hypersurface which is not a cone over a smaller dimensional cubic, J. Kollár [18, Theorem 1.2] proved that having a k -rational point is equivalent to being unirational over k . The nonsingular cubic hypersurfaces in $\mathbb{P}_{\mathbb{C}}^4$, which are in particular unirational, are proven by C. H. Clemens and P. Griffiths to be nonrational varieties [12, Theorem 13.12]. However, it is not known whether it is stably rational or not, therefore it would be interesting to study this particular instance. Unfortunately we are not able to answer Question 3.7 for this difficult case. In the higher dimensions questions of the rationality and stable rationality of nonsingular cubic hypersurfaces are open in general.

4. The Class of a Quadric in $K_0(\text{Var}_k)$

In this section we will consider the quadrics, i.e. degree two hypersurfaces, since these hypersurfaces would be the next natural example after hyperplanes to look for the answer to Question 3.7.

THEOREM 4.1. ([13, Theorem 1.11]) *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}_k^n$ be an irreducible quadric hypersurface. Then X is rational if and only if it has a smooth k -rational point.*

By Remark 2.10, for a smooth quadric X , we have X stably rational if and only if $[X] \equiv 1$ modulo \mathbb{L} over a field of characteristic zero. This together with Theorem 4.1 and Proposition 2.11 give the positive answer to Question 3.7: for a smooth quadric X , one has $X(k) \neq \emptyset$ if and only if $[X] \equiv 1$ modulo \mathbb{L} . We like to give a proof to show that for a not necessarily smooth quadric hypersurface over a characteristic zero field, the answer to the Question 3.7 is positive. In order to prove $X(k) \neq \emptyset$ implies $[X] \equiv 1 \pmod{\mathbb{L}}$, we aim to write down some explicit affine fibration to obtain the class of X in $K_0(\text{Var}_k)$ in the form $1 + \mathbb{L} \cdot [Y]$, where $[Y] \in K_0(\text{Var}_k)$. We will use the following well known lemma from linear algebra, which is about the nondegenerate quadratic forms with a nontrivial solution over the base field of characteristic different than two, first for writing down such an affine fibration for a smooth quadric, and then later in Section 7 for proving Theorem 7.1.

LEMMA 4.2. *Let k be a field with $\text{char}(k) \neq 2$, and let $X = V(q)$ be a smooth quadric hypersurface in \mathbb{P}_k^n . Let $x \in X(k)$ be a point. Then we can choose coordinates in \mathbb{P}_k^n such that*

$$x_i(x) = 0, \text{ for all } i \neq 1,$$

and

$$q(x_0, \dots, x_n) = x_0x_1 + q'(x_2, \dots, x_n),$$

where $q' \in k[x_2, \dots, x_n]$ is a quadratic form.

PROOF. Denote by B_q the bilinear form associated to q . Let $v \in k^{\oplus n+1} \setminus \{0\}$ be the corresponding vector to the point x , then we have $B_q(v, v) = 0$. Now we claim that we can choose a vector $w \in k^{\oplus n+1} \setminus \{0\}$ such that

$$B_q(v, w) \neq 0 \text{ and } B_q(w, w) = 0.$$

Indeed, there exists some $w \in k^{\oplus n+1} \setminus \{0\}$ such that $B_q(v, w) \neq 0$, since q is nondegenerate. Moreover we can always choose a vector w among those that satisfies $B_q(v, w) \neq 0$ in a way that it also satisfies $B_q(w, w) = 0$ as we see below:

Let $w' \in k^{\oplus n+1} \setminus \{0\}$ be a vector with $B_q(v, w') \neq 0$ and $B_q(w', w') \neq 0$.

Ansatz: $w = w' + \alpha v$, for some $\alpha \in k$.

Note that then w is nonzero, because $B_q(w', w') \neq 0$ implies that $w' \neq \gamma v$ for any $\gamma \in k$. Thus

$$B_q(v, w) = B_q(v, w') + \alpha B_q(v, v) = B_q(v, w') \neq 0, \text{ and}$$

$$\begin{aligned}
B_q(w, w) &= B_q(w' + \alpha v, w' + \alpha v) \\
&= B_q(w', w' + \alpha v) + \alpha B_q(v, w' + \alpha v) \\
&= B_q(w', w') + \alpha B_q(w', v) + \alpha B_q(v, w') + \alpha^2 B_q(v, v) \\
&= B_q(w', w') + 2\alpha B_q(w', v).
\end{aligned}$$

Thus, by taking $\alpha = \frac{-B_q(w', w')}{2B_q(w', v)}$, we get a $w \in k^{\oplus n+1} \setminus \{0\}$ with

$$B_q(w, w) = 0.$$

Note that here we have $\dim \langle w, v \rangle = 2$. To see this, let $\alpha w + \beta v = 0$ with $\alpha, \beta \in k$. Then we get

$$\begin{aligned}
0 &= B_q(\alpha w + \beta v, \alpha w + \beta v) \\
&= \alpha B_q(w, \alpha w + \beta v) + \beta B_q(v, \alpha w + \beta v) \\
&= \alpha^2 B_q(w, w) + \alpha \beta B_q(w, v) + \beta \alpha B_q(v, w) + \beta^2 B_q(v, v) \\
&= 2\alpha \beta B_q(w, v).
\end{aligned}$$

Thus $\alpha = 0$, and so $\beta = 0$, which proves that $\langle w, v \rangle$ is of dimension 2. Let us define

$$\langle w, v \rangle^\perp := \{u \in k^{\oplus n+1} \mid B_q(u, w) = B_q(u, v) = 0\}.$$

We will show that

$$k^{\oplus n+1} \cong \langle w, v \rangle \oplus \langle w, v \rangle^\perp.$$

Claim: There exists $u' \in \langle w, v \rangle^\perp$ such that $u = \alpha w + \beta v + u'$ for all $u \in k^{\oplus n+1}$.

Let $B_q(u, w) = \gamma$ and $B_q(u, v) = \delta$. We have

$$\begin{aligned}
B_q(u, w) &= \alpha B_q(w, w) + \beta B_q(v, w) + B_q(u', w) \\
\gamma &= \beta B_q(v, w) + B_q(u', w). \\
B_q(u, v) &= \alpha B_q(v, v) + \beta B_q(w, v) + B_q(u', v) \\
\delta &= \alpha B_q(w, v) + B_q(u', v).
\end{aligned}$$

Indeed, by taking $\alpha = \frac{\delta}{B_q(w, v)}$ and $\beta = \frac{\gamma}{B_q(w, v)}$, we get $u' \in \langle w, v \rangle^\perp$.

Hence the following is an isomorphism of k -vector spaces:

$$\begin{array}{ccc}
k^{\oplus n+1} & \longrightarrow & \langle w, v \rangle \oplus \langle w, v \rangle^\perp \\
u & \longmapsto & \alpha w + \beta v + u'.
\end{array}$$

Let $\Gamma := \{w, v, e'_2, \dots, e'_n\}$ be the new basis for $k^{\oplus n+1}$, where $\{e'_2, \dots, e'_n\}$ is some basis for the subspace $\langle w, v \rangle^\perp$. Denote by $w^*, v^*, e'^*_2, \dots, e'^*_n$ the new coordinates corresponding to the basis Γ , then we have

$$q(w^*, v^*, e'^*_2, \dots, e'^*_n) = w^* v^* + q'(e'^*_2, \dots, e'^*_n).$$

□

REMARK 4.3. With the new coordinates from Lemma 4.2, the point x corresponds to the point $[0 : 1 : 0 : \dots : 0]$. This will be a convenient choice for Proposition 7.1.

For ground fields of characteristic not equal to 2, above lemma and the well known fact that every quadratic form can be diagonalized will be enough for our purposes. On the other hand the case of ground fields of characteristic 2 is different, in the sense that there exist quadratic forms which cannot be diagonalized, and moreover a degenerate quadratic form may as well have no nontrivial solution unlike the cases where $\text{char}(k) \neq 2$. However this case has also been studied thoroughly by the means of classification. The following lemma provides a decomposition for any quadratic form over fields of characteristic 2 that will allow us to include the characteristic 2 fields, in Theorem 4.5- (1).

LEMMA 4.4. (cf. [17, §2, Proposition 2.4]) *Let k be a field of characteristic 2, and let q be a quadratic form over a k -vector space of dimension $n + 1$. Then we can choose coordinates such that*

$$q(x) = \sum_{i=1}^r x_i y_i + \tilde{q}(z_1, \dots, z_s)$$

where $x = (x_1, y_1, \dots, x_r, y_r, z_1, \dots, z_s, t_1, \dots, t_m)$ with $2r + s + m = n + 1$, $r, s, m \geq 0$ and $\tilde{q}(z_1, \dots, z_s) \in k[z_1, \dots, z_s]$ is a quadratic form which has no nontrivial solution over k .

Now we state the following theorem for the quadric hypersurfaces and give a proof for it, considering the smooth and singular cases separately, using simply the projections from the k -rational points of X for one side of the implication and the result of Larsen and Lunts for the other.

THEOREM 4.5. *Let k be a field. Let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree 2 where $n \geq 2$.*

- (1) *If $X(k) \neq \emptyset$ then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$.*
- (2) *Assume $\text{char}(k) = 0$. If $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$ then $X(k) \neq \emptyset$.*

PROOF. (2): Any degenerate quadratic form over a field of characteristic not equal to 2 has nontrivial solutions, therefore in the case that X is singular, it always has a k -rational point. If $[X] \equiv 1 \pmod{\mathbb{L}}$ for X smooth, then we know by Proposition 2.11 that $X(k) \neq \emptyset$.

(1): Now let us consider a quadric $X = V(q)$ where q is a quadratic form. We will examine the case $\text{char}(k) = 2$ separately, thus for now we assume that $\text{char}(k) \neq 2$. First we consider the case X is smooth, that is q is nondegenerate. We make the change of coordinates as given in Lemma 4.2, and denote by x_0, \dots, x_n these new coordinates. Let

$$U_0 := \{[x_0 : \dots : x_n] \in \mathbb{P}_k^n \mid x_0 \neq 0\} \subset \mathbb{P}_k^n,$$

then

$$U_0 \cap X = V\left(\frac{x_1}{x_0} + q'\left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)\right) \cong \mathbb{A}_k^{n-1},$$

since we have

$$\frac{k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]}{(\frac{x_1}{x_0} + q'(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}))} \cong k[\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}].$$

Let $Y := V(q') \subset \mathbb{P}_k^{n-2}$. Now let us consider the Zariski locally trivial affine fibration

$$\begin{aligned} p : \mathbb{P}_k^n \setminus (U_0 \cup \{[0 : 1 : 0 : \cdots : 0]\}) &\longrightarrow \mathbb{P}_k^{n-2} \\ [x_0 : \cdots : x_n] &\longmapsto [x_2 : \cdots : x_n]. \end{aligned}$$

Then the following is a cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus (U_0 \cup \{[0 : 1 : 0 : \cdots : 0]\}) & \longrightarrow & \mathbb{P}_k^{n-2} \\ \uparrow & & \uparrow \\ X \setminus ((X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\}) & \longrightarrow & Y \end{array}$$

Hence

$$p|_X : X \setminus ((X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\}) \longrightarrow Y$$

is also Zariski locally trivial with fibres isomorphic to \mathbb{A}_k^1 . Then by Remark 2.4 we have

$$\begin{aligned} [X] &= [X \setminus ((X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\})] + \\ &\quad [(X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\}] \\ &= \mathbb{L} \cdot [Y] + [X \cap U_0] + 1 \\ &= 1 + \mathbb{L} \cdot [Y] + \mathbb{L}^{n-1} \end{aligned}$$

and $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$.

Now let us consider the case where X is singular. Since one can diagonalize every quadratic form, after a change of coordinates we can write $q(x_0, \dots, x_n) = a_0x_0^2 + \cdots + a_rx_r^2$, with $r < n$ since q is degenerate. Thus we have

$$\mathbf{P} := \{[0 : \cdots : 0 : x_{r+1} : \cdots : x_n] \mid [x_{r+1} : \cdots : x_n] \in \mathbb{P}_k^{n-r-1}\} \subset X = V(q)$$

Let $Y := V(a_0x_0^2 + \cdots + a_rx_r^2) \subset \mathbb{P}_k^r$. Consider the projection

$$\begin{aligned} p : \mathbb{P}_k^n \setminus \mathbf{P} &\longrightarrow \mathbb{P}_k^r \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_r]. \end{aligned}$$

Observe that $x \in X \setminus \mathbf{P}$ if and only if $p(x) \in Y$: indeed $p(x) \in Y$ if and only if $x \notin \mathbf{P}$ and $q(x) = 0$. Therefore the following is a cartesian diagram

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus \mathbf{P} & \longrightarrow & \mathbb{P}_k^r \\ \uparrow & & \uparrow \\ X \setminus \mathbf{P} & \longrightarrow & Y \end{array}$$

and the projection p is Zariski locally trivial, hence we see that the map $X \setminus \mathbf{P}$ is also Zariski locally trivial fibration with fibres isomorphic to \mathbb{A}_k^{n-r} ,

and by Remark 2.4 we obtain

$$\begin{aligned} [X] &= [X \setminus \mathbf{P}] + [\mathbf{P}] \\ &= \mathbb{L}^{n-r} \cdot [Y] + [\mathbb{P}_k^{n-r-1}] \\ &= 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-r-1} + \mathbb{L}^{n-r} \cdot [Y]. \end{aligned}$$

Thus $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(Var_k)$. This concludes the proof over fields of characteristic not equal to 2.

Now assume that $\text{char}(k) = 2$: then by Lemma 4.4 we have

$$q(x) = \sum_{i=1}^r x_i y_i + \tilde{q}(z_1, \dots, z_s)$$

where $V(\tilde{q}) \subset \mathbb{P}_k^{s-1}$ has no k -rational points. First of all, let us note that assuming $X = V(q)$ has a k -rational point, we eliminate the possibility that $r = 0$ and $n = s - 1$. There are in fact two possible cases:

Case 1: $r \geq 1$. This means that q can be written of the form $x_1 y_1 + q'(x')$ for some quadratic form $q' \in k[x']$ where $x' = (x_2, y_2, \dots, x_r, y_r, z_1, \dots, z_s, t_1, \dots, t_m)$ with the coordinates from Lemma 4.4. Then we can proceed the same way we did in the case of a smooth quadric over a field of characteristic not equal to 2, i.e. we can obtain a Zariski locally trivial fibration by projecting from $U_0 \cup \{[0 : 1 : 0 : \cdots : 0]\}$ to \mathbb{P}_k^{n-2} , where

$$U_0 := \{[x_0 : \cdots : x_n] \in \mathbb{P}_k^n \mid x_0 \neq 0\} \subset \mathbb{P}_k^n.$$

Case 2: $r = 0$ and $m = n - (s + 1) > 0$. Then $q(x) = \tilde{q}(z_1, \dots, z_s) \in k[z_1, \dots, z_s, t_1, \dots, t_m]$ and

$$\mathbf{R} := \{[0 : \cdots : 0 : t_1 : \cdots : t_m] \in \mathbb{P}_k^n \mid [t_1 : \cdots : t_m] \in \mathbb{P}_k^{m-1}\} \subset X.$$

Therefore we can proceed the same way we did in the case of a singular quadric over a field of characteristic not equal to 2, i.e. we can project from \mathbf{R} to \mathbb{P}_k^{s-1} . Then we get a Zariski locally trivial fibration $X \setminus \mathbf{R} \rightarrow V(\tilde{q})$ whose fibers are isomorphic to \mathbb{A}_k^m . We conclude by using Remark 2.4 for this fibration. Hence this completes the proof of (1). \square

COROLLARY 4.6. *For quadric hypersurfaces over a field of characteristic zero Question 3.7 has a positive answer.*

5. The Class of a Union of Hyperplanes

For a hypersurface of degree greater than two, the simplest case to consider is the union of hyperplanes. Such a variety always has k -rational points. Therefore in this case Question 3.7 asks if the class $[X]$ is always equivalent to 1 modulo \mathbb{L} . We prove that the answer is yes, using the projection from the intersection of all of the hyperplanes, in order to write the class of the given hypersurface in the form $1 + \mathbb{L} \cdot [Y]$, where $[Y] \in K_0(\text{Var}_k)$.

THEOREM 5.1. *Let k be a field, and let $X := V(h_1 \cdots h_d) \subset \mathbb{P}_k^n$ be a hypersurface of degree $d \leq n$ where $h_i \in k[x_0, \dots, x_n]$ are homogeneous polynomials of degree 1, for $1 \leq i \leq d$, then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$.*

PROOF. Let us first note that one can assume $\{h_i\}_{i=1}^d$ to be distinct polynomials: otherwise we would have $\{h_{ij}\}_{i,j=1}^{e,f}$ with $e, f \leq d$ such that $h_{i1} = \cdots = h_{if}$ for all $1 \leq i \leq e$, and $h_{i1} \neq h_{k1}$ for $i \neq k$. Hence we would get

$$[X] = [X_{\text{red}}] = [V(\prod_{i=1}^e h_{i1})]$$

by Remark 2.3.

Now let $Y := \bigcap_{i=1}^d V(h_i)$, let r be the codimension of Y in \mathbb{P}_k^n . Since Y is an intersection of hyperplanes, it is isomorphic to the projective space \mathbb{P}_k^{n-r} . Note that $n - r \geq 0$ since we have $r \leq d \leq n$. Therefore applying a coordinate change we get

$$Y = \bigcap_{i=1}^d V(h_i) = \{[0 : \cdots : 0 : x_r : \cdots : x_n] \mid [x_r : \cdots : x_n] \in \mathbb{P}_k^{n-r}\} \cong \mathbb{P}_k^{n-r}.$$

Since $h_i(x) = 0$ for all $x \in Y$ and for all $i = 1, \dots, d$, we find that $h_i \in k[x_0, \dots, x_{r-1}]$ for all $i = 1, \dots, d$. Thus we get $f := h_1 \cdots h_d \in k[x_0, \dots, x_{r-1}]$. Let $Z := V(f) \subset \mathbb{P}_k^{r-1}$, and let us consider the Zariski locally trivial affine fibration

$$\begin{aligned} p : \mathbb{P}_k^n \setminus Y &\longrightarrow \mathbb{P}_k^{r-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_{r-1}]. \end{aligned}$$

Note that then we have $x \in X \setminus Y$ if and only if $p(x) \in Z$: indeed $p(x) \in Z$ if and only if $x \notin Y$ and $f(x) = 0$. Hence the following is a cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus Y & \xrightarrow{p} & \mathbb{P}_k^{r-1} \\ \uparrow & & \uparrow \\ X \setminus Y & \longrightarrow & Z \end{array}$$

Thus

$$p|_X : X \setminus Y \longrightarrow Z$$

is also Zariski locally trivial with fibres isomorphic to \mathbb{A}_k^{n-r+1} . Thus we get

$$[X \setminus Y] = \mathbb{L}^{n-r+1} \cdot [Z].$$

Hence by Remark 2.4 we obtain

$$\begin{aligned} [X] &= [X \setminus Y] + [Y] \\ &= \mathbb{L}^{n-r+1} \cdot [Z] + [\mathbb{P}_k^{n-r}] \\ &= 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-r} + \mathbb{L}^{n-r+1} \cdot [Z] \end{aligned}$$

and $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$. \square

COROLLARY 5.2. *Let X be as in Theorem 5.1, then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$ if and only if $X(k) \neq \emptyset$.*

PROOF. As we noted above, any point of the form

$$[0 : \cdots : 0 : x_r : \cdots : x_n] \in \mathbb{P}_k^n$$

is a point of X . Hence X has always k -rational points. \square

After checking union of hyperplanes, next example we consider will be the hypersurfaces which become a union of hyperplanes after a finite Galois base change. Below we state and prove that also for such hypersurfaces the answer to Question 3.7 is positive. As in the proof of Theorem 5.1, we consider the intersection of all hyperplanes forming the hypersurface after the base change, more precisely we consider the quotient variety defined by this intersection modulo the Galois group of the extension. This is a closed subvariety of the given hypersurface over k . Using the Hilbert 90 theorem, we show that this quotient variety is a linear subspace of \mathbb{P}_k^n . Then we project from this quotient variety and conclude with the proof.

THEOREM 5.3. *Let L be a finite Galois extension of k , let X be a hypersurface in \mathbb{P}_k^n of degree $d \leq n$ such that $X \times_k L$ is a union of d hyperplanes over L , that is $X \times_k L = V(h_1 \cdots h_d)$ where $h_i \in L[x_0, \dots, x_n]$ are homogeneous of degree 1 for $1 \leq i \leq d$. Then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$, and $X(k) \neq \emptyset$.*

PROOF. As in Theorem 5.1, we may assume that the polynomials h_i 's are distinct.

Let $Y := \bigcap_{i=1}^d V(h_i)$ in \mathbb{P}_L^n , and let $G := \text{Gal}(L/k)$. Then G acts on $X \times_k L$, and also on Y since one has $g \cdot y \in Y$ for all $y \in Y$. Therefore we see that $Y/G \subset X$ is a closed subvariety.

Claim. Y/G is a linear subspace of \mathbb{P}_k^n .

In order to prove this claim, let us consider the L -vector subspace

$$V := L \langle h_1, \dots, h_d \rangle \subset L^{n+1}.$$

Since G acts on $V(h_1 \cdots h_d) = X \times_k L$, it also acts on V . Therefore one can use Lemma 5.4 below to deduce that there exists homogeneous polynomials $\{h'_i\}_{i=1}^r$ in $k[x_0, \dots, x_n]$ of degree 1 such that $L \langle h'_1, \dots, h'_r \rangle = V$, where

$r := \dim_L V$. Hence, $Y/G = \bigcap_{i=1}^r V(h'_i)$ and $Y = (Y/G) \times_k L$. Now that Y/G is a linear subspace of \mathbb{P}_k^n of codimension r , we can choose the coordinates so that we have

$$Y/G = \{[0 : \cdots : 0 : x_r : \cdots : x_n] \mid [x_r : \cdots : x_n] \in \mathbb{P}_k^{n-r}\}.$$

Consider the projection map

$$\begin{aligned} p : \mathbb{P}_k^n \setminus (Y/G) &\longrightarrow \mathbb{P}_k^{r-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_{r-1}]. \end{aligned}$$

Observe that

$$h_i \in L[x_0, \dots, x_{r-1}]$$

and

$$f := \prod_{i=1}^d h_i \in k[x_0, \dots, x_{r-1}]$$

in a similar manner to the polynomials h_i of Theorem 5.1 (see proof of Theorem 5.1). Let $Z := V(f) \subset \mathbb{P}_k^{r-1}$. With this setting we obtain that $x \in X \setminus (Y/G)$ if and only if $p(x) \in Z$ since for all $x \in \mathbb{P}_k^n \setminus Y/G$ one has $f(x) = 0$ if and only if $f(p(x)) = 0$. Therefore the following diagram

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus (Y/G) & \xrightarrow{p} & \mathbb{P}_k^{r-1} \\ \uparrow & & \uparrow \\ X \setminus (Y/G) & \longrightarrow & Z \end{array}$$

is cartesian, $p|_X : X \setminus (Y/G) \longrightarrow Z$ is Zariski locally trivial affine fibration with fibres isomorphic to \mathbb{A}_k^{n-r+1} . Thus we are now again in the same situation as in Theorem 5.1, i.e., $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$. Moreover $X(k) \neq \emptyset$ since $\emptyset \neq (Y/G)(k) \subset X$ is a closed subvariety. □

LEMMA 5.4 (Hilbert 90). *Let k be a field, and let L be a finite Galois extension of k . We denote by $G := \text{Gal}(L/k)$ the Galois group of the extension L/k . Let $V \subset L^n$ be a G -invariant L -vector subspace, i.e., $g(V) \subset V$ for all $g \in G$, then there exists a k -vector space $V' \subset k^n$ such that $V' \otimes_k L = V$ where $V' \otimes_k L$ is considered as an L -vector subspace of L^n by the extension of the scalars.*

PROOF. Let $r := \dim_L V$, and let $\{t_i\}_{i=1}^r$ be an L -basis of V . Since $g(V) \subset V$ for all $g \in G$, G acts on V , as below

$$g \cdot t_i = \sum_{j=1}^r m_{ji}(g) t_j$$

where $m_{ij}(g) \in L$ not all simultaneously zero, for all t_i . We need to prove that there exists an L -basis $\{t'_i\}_{i=1}^r$ of V such that $g \cdot t'_i = t'_i$ for all i , because this means that $\{t'_i\}_{i=1}^r$ consist a k -basis for a vector space V' such that

$V' \otimes_k L = V$. For this aim, let us define the map

$$\begin{aligned} \alpha : G &\longrightarrow GL_r(L) \\ g &\longmapsto \alpha_g := (m_{ji}(g))_{i,j}^{-1}. \end{aligned}$$

Here α_g is indeed in $GL(V)$, since $\{t_i\}_{i=1}^r$ is a basis for V .

Recall that a 1-cocycle with values in $GL(V)$ is a map $\rho : G \longrightarrow GL(V)$ such that $\rho(g_1 g_2) = \rho(g_1) g_1(\rho(g_2))$ for all $g_1, g_2 \in G$.

Claim: $\alpha : G \longrightarrow GL_r(L)$ is a 1-cocycle. Indeed, for all $g_1, g_2 \in G$ we have

$$g_1 g_2 \cdot t_i = g_1 \cdot (g_2 \cdot t_i) = \sum_{j=1}^r \sum_{k=1}^r g_1(m_{ki}(g_2)) m_{jk}(g_1) t_j$$

Thus we have

$$\begin{aligned} \alpha_{g_1 g_2} &= ((m_{ji}(g_1 g_2))_{i,j})^{-1} \\ &= [(g_1(m_{ji}(g_2))(m_{ji}(g_1)))_{i,j}]^{-1} \\ &= ((m_{ji}(g_1))_{i,j})^{-1} (g_1(m_{ji}(g_2))_{i,j})^{-1} \\ &= \alpha_{g_1} g_1(\alpha_{g_2}). \end{aligned}$$

By the Hilbert 90 theorem [29, Chapter X, Proposition 3], we know that

$$H^1(G, GL_r(L)) = \{1\},$$

i.e., every 1-cocycle with values in $GL_r(L)$ is cohomologous to the trivial 1-cocycle (that maps every element of G to $id \in GL_r(L)$). Hence α is cohomologous to 1, i.e. there exists a $B \in GL_r(L)$ such that $\alpha_g = B g(B^{-1})$ for all $g \in G$. Now the following claim will complete the proof of the lemma: Claim: $\{B^{-1} \cdot t_i\}_{i=1}^r$ is an L -basis of V such that $g(B^{-1} \cdot t_i) = B^{-1} \cdot t_i$ for all $i = 1, \dots, r$.

Indeed, we have

$$\begin{aligned} g(B^{-1} \cdot t_i) &= g(B^{-1}) g \cdot t_i \\ &= g(B)^{-1} \alpha_g^{-1} \cdot t_i \\ &= g(B)^{-1} g(B) B^{-1} \cdot t_i \\ &= B^{-1} \cdot t_i. \end{aligned}$$

□

COROLLARY 5.5. *For hypersurfaces of the type given in Theorem 5.1 and 5.3, Question 3.7 has a positive answer.*

6. Cubic Hypersurfaces

In this section we will consider the case of degree three hypersurfaces for Question 3.7. Therefore we are only interested in the hypersurfaces that live in at least three dimensional projective space. Note that over an algebraically closed field k , it is known that a smooth cubic surface is rational [23, Chapter IV, Theorem 24.1]. For a smooth projective hypersurface X over a field of characteristic zero, (stable) rationality implies that the class $[X]$ is equivalent to 1 modulo \mathbb{L} in $K_0(Var_k)$, by Proposition 2.9. On the other hand there are smooth cubic hypersurfaces that are not rational, like smooth cubic threefolds which are proven to be non-rational by Clemens and Griffiths in their paper [12]. It is still unknown whether smooth cubic threefolds are stably rational or not. If an irreducible projective cubic hypersurface that is not a cone over a cubic of lower dimension has a k -rational singular point, then it is rational [13, Chapter 1, Section 5, Example 1.28]. However in the singular case rational varieties do not necessarily have class 1 modulo \mathbb{L} , but it is possible to show that a singular cubic hypersurface with a rational singular point, actually has this property. This is what we are going to prove next.

THEOREM 6.1. *Let k be a C1 field, and let X be a hypersurface of degree 3 in \mathbb{P}_k^n where $n \geq 3$. Denote by $X_{sing}(k)$ the set of k -rational points of the singular locus of X . If $X_{sing}(k) \neq \emptyset$, then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(Var_k)$.*

PROOF. Let $x \in X_{sing}(k)$. After a change of coordinates, we get $x = [0 : \cdots : 0 : 1]$. Let $X = V(f)$. Since $x = [0 : \cdots : 0 : 1] \in X(k)$, f has the following form with these new coordinates

$$f(x_0, \dots, x_n) = x_n^2 f_1(x_0, \dots, x_{n-1}) + x_n f_2(x_0, \dots, x_{n-1}) + f_3(x_0, \dots, x_{n-1})$$

where f_i are homogeneous polynomials of degree i , $i = 1, 2, 3$. Moreover x is a singular point of X , i.e.

$$\frac{\partial f}{\partial x_i} \Big|_x = 0, \text{ for all } 0 \leq i \leq n.$$

Let us note that we have in particular

$$\frac{\partial f_1}{\partial x_i} \Big|_x = 0, \text{ for all } 0 \leq i \leq n-1.$$

Hence we get $f_1 = 0$. Now let us consider the following Zariski locally trivial affine fibration:

$$\begin{aligned} \pi : \quad \mathbb{P}_k^n \setminus \{x\} &\longrightarrow \mathbb{P}_k^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_0 : \cdots : x_{n-1}]. \end{aligned}$$

Note that

$$\pi^{-1}(p) = \{[p_0 : \cdots : p_{n-1} : \gamma] \mid \gamma \in k\} \cong \mathbb{A}_k^1$$

and that $\pi^{-1}(p) \cup \{x\} \cong \mathbb{P}_k^1$ for all $p = [p_0 : \cdots : p_{n-1}] \in \mathbb{P}_k^{n-1}$. Let us denote by π_X the restriction of π to X . Therefore we get that

$$\begin{aligned} [X] &= [X \setminus \{x\}] + [\{x\}] \\ &= 1 + [\pi_X^{-1}(\mathbb{P}_k^{n-1})] \end{aligned} \quad (6.0.1)$$

by Remark 2.4. For any $p \in \mathbb{P}_k^{n-1}$ we have one of the following possibilities for the fibre of π_X at p

$$\pi_X^{-1}(p) = \begin{cases} y \in X \setminus \{x\} \\ \pi^{-1}(p) \\ \emptyset \end{cases}$$

First of all, let us observe that

$$\begin{aligned} Y &:= \{p \in \mathbb{P}_k^{n-1} \mid \pi_X^{-1}(p) = y \in X \setminus \{x\}\} \cong \pi_X^{-1}(Y) \\ &= \{p \in \mathbb{P}_k^{n-1} \mid \pi_X^{-1}(p) = [p_0 : \cdots : p_{n-1} : -f_3(p)/f_2(p)]\} \\ &= \{p \in \mathbb{P}_k^{n-1} \mid f_2(p) \neq 0\} \\ &= \mathbb{P}_k^{n-1} \setminus V(f_2). \end{aligned}$$

Note that k being a $C1$ field assures that

$$V(f_2)(k) \neq \emptyset,$$

which implies by the Theorem 4.5-(1) that $[V(f_2)] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$. Hence

$$\begin{aligned} [\pi_X^{-1}(Y)] &= [Y] = [\mathbb{P}_k^{n-1} \setminus V(f_2)] \\ &= [\mathbb{P}_k^{n-1}] - [V(f_2)] \\ &\equiv 0 \pmod{\mathbb{L}} \end{aligned} \quad (6.0.2)$$

Let us denote by Z the set of points of which the pre-images under π is completely contained in X :

$$\begin{aligned} Z &:= \{p \in \mathbb{P}_k^{n-1} \mid \pi^{-1}(p) \subset X \setminus \{x\}\} \\ &= \{p \in \mathbb{P}_k^{n-1} \mid \gamma f_2(p) + f_3(p) = 0, \forall \gamma \in k\} \\ &= V(f_2, f_3) \end{aligned}$$

Hence the diagram

$$\begin{array}{ccc} \mathbb{P}_k^n \setminus \{x\} & \xrightarrow{\pi} & \mathbb{P}_k^{n-1} \\ \uparrow & & \uparrow \\ \pi^{-1}(Z) & \xrightarrow{\pi_X} & Z \end{array}$$

is cartesian, and $\pi_X^{-1}(Z) \rightarrow Z$ is a Zariski locally trivial \mathbb{A}_k^1 -fibration. This yields by Remark 2.4 that

$$[\pi_X^{-1}(Z)] = [\pi^{-1}(Z)] = \mathbb{L} \cdot [V(f_2, f_3)] \quad (6.0.3)$$

Thus we obtain

$$\begin{aligned}
[X] &= 1 + [\pi_X^{-1}(\mathbb{P}_k^{n-1})] \quad \text{by Equation 6.0.1,} \\
&= 1 + [\pi_X^{-1}(Y) \sqcup \pi_X^{-1}(Z)] \\
&= 1 + [Y] + \mathbb{L} \cdot [V(f_2, f_3)] \quad \text{by Equation 6.0.3,} \\
&\equiv 1 \pmod{\mathbb{L}} \quad \text{by Equation 6.0.2 .}
\end{aligned}$$

in $K_0(\text{Var}_k)$. □

COROLLARY 6.2. *For cubic hypersurfaces over k with a singular k -rational point, Question 3.7 is positively answered.*

7. Union of Two Quadric Hypersurfaces

In this section we consider a particular example of quartic hypersurfaces, namely ones in \mathbb{P}_k^n for any $n \geq 4$, which consist of the union of two quadric hypersurfaces, one of which is smooth. We need to assume that k is algebraically closed to conclude with the proof. We will compute the class in $K_0(\text{Var}_k)$ modulo the class \mathbb{L} . For this we prove that the class of the intersection of the two quadrics which form the quartic is equivalent to the class of a cubic hypersurface in \mathbb{P}_k^{n-1} , modulo \mathbb{L} . We need the assumption k is algebraically closed in order to guarantee that this cubic hypersurface has a k -rational singular point for any $n \geq 5$ and thus has class equivalent to 1 modulo \mathbb{L} , by Theorem 6.1. For $n = 4$ we get in general a smooth cubic surface in \mathbb{P}_k^3 , but this is rational which in this case implies that we are done, by Proposition 2.9.

THEOREM 7.1. *Let k be an algebraically closed field of characteristic zero, let X be a union of two quadric hypersurfaces $Q_1, Q_2 \subset \mathbb{P}_k^n$, where Q_1 is smooth and $n \geq 4$. Then $[X] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$.*

PROOF. If $Q_1 = Q_2$, then $[Q_1 \cup Q_2] = [Q_1 \cup Q_1] = [Q_1]$. Hence it reduces to the case of Theorem 4.5, and we are done. For distinct Q_1, Q_2 we have

$$[X] = [Q_1 \cup Q_2] = [Q_1] + [Q_2] - [Q_1 \cap Q_2]$$

in $K_0(\text{Var}_k)$. Since k is algebraically closed, both Q_1 and Q_2 have k -rational points. Hence, by Theorem 4.5 we get

$$[Q_i] \equiv 1 \pmod{\mathbb{L}}, \quad i = 1, 2, \quad (7.0.1)$$

and therefore $[X] \equiv 1 \pmod{\mathbb{L}}$ if and only if $[Q_1 \cap Q_2] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$. Let

$$Q_i := V(q_i), \quad i = 1, 2.$$

Since we are over an algebraically closed field, the intersection $Q_1 \cap Q_2$ also has a k -rational point x . Since Q_1 is smooth, we can make a change of coordinates in order to get $q_1(x_0, \dots, x_n) = x_0x_1 - h(x_2, \dots, x_n)$. To do this, we apply Lemma 4.2 to q_1 and x . Hence we get coordinates in which q_1 is of the desired form, and the point x becomes the point $[0 : 1 : 0 : \dots : 0]$. Now since $[0 : 1 : 0 : \dots : 0] \in Q_2$, the monomial x_1^2 does not appear in q_2 . Thus we can write

$$q_2(x_0, \dots, x_n) = x_0L_0(x_0, x_2, \dots, x_n) + x_1L_1(x_0, x_2, \dots, x_n) + R(x_2, \dots, x_n)$$

where L_0, L_1, R are homogeneous polynomials, L_0, L_1 are linear and R is of degree 2. Let $U_0 := \{[x_0 : \dots : x_n] \mid x_0 \neq 0\}$. We will calculate the class of the intersection of $Q_1 \cap Q_2$ with U_0 . Let $Q_1^{\{x_0 \neq 0\}} := Q_1 \cap U_0 = V(q_1|_{U_0})$. We have

$$q_1|_{U_0}\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = \frac{x_1}{x_0} - h\left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

and $Q_1^{\{x_0 \neq 0\}} \cong \mathbb{A}_k^{n-1}$ where the isomorphism is given by

$$\begin{aligned} \varphi : Q_1^{\{x_0 \neq 0\}} &\longrightarrow \mathbb{A}_k^{n-1} \\ \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) &\longmapsto \left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) \end{aligned}$$

Let us now consider the intersection $Q_1^{\{x_0 \neq 0\}} \cap Q_2$. Via the isomorphism φ , it is defined by the following polynomial:

$$g\left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) := L_0\left(1, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) + h\left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)L_1\left(1, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) + R\left(\frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

i.e.,

$$\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2) = V(g) \subset \mathbb{A}_k^{n-1}.$$

We embed $\mathbb{A}_k^{n-1} \subset \mathbb{P}_k^{n-1}$ as $\{y_1 \neq 0\}$ with homogeneous coordinates $\{y_1, \dots, y_n\}$ for \mathbb{P}_k^{n-1} . Then the closure

$$Y := \overline{\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2)} \subset \mathbb{P}_k^{n-1}$$

is defined by the homogenization of g :

$$\bar{g}(y_1, \dots, y_n) = y_1^2 L_0(y_1, \dots, y_n) + h(y_2, \dots, y_n) L_1(y_1, \dots, y_n) + y_1 R(y_2, \dots, y_n).$$

Thus

$$\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2) = V(g) = Y \setminus (Y \cap V(y_1)),$$

and we get

$$[\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2)] = [Y] - [Y \cap V(y_1)]. \quad (7.0.2)$$

The intersection $Y \cap V(y_1)$ is the vanishing locus of

$$h(y_2, \dots, y_n) L_1(0, y_2, \dots, y_n).$$

Hence

$$\begin{aligned} [Y \cap V(y_1)] &= [V(h L_1) \cap V(y_1)] \\ &= [V(h) \cap V(y_1)] + [V(L_1) \cap V(y_1)] - [V(h) \cap V(L_1) \cap V(y_1)] \end{aligned}$$

in $K_0(\text{Var}_k)$. Here $V(h) \cap V(y_1)$ is a quadric hypersurface and $V(L_1) \cap V(y_1)$ is a hyperplane in \mathbb{P}_k^{n-2} . Since $n - 2 \geq 2$ by the assumption, we get

$$[V(h) \cap V(y_1)], [V(L_1) \cap V(y_1)] \equiv 1 \pmod{\mathbb{L}}$$

in $K_0(\text{Var}_k)$. Hence

$$[Y \cap V(y_1)] \equiv 2 - [V(h) \cap V(L_1) \cap V(y_1)] \pmod{\mathbb{L}} \quad (7.0.3)$$

in $K_0(\text{Var}_k)$. Now let us examine

$$Q_1^{\{x_0=0\}} \cap Q_2 := Q_1 \cap Q_2 \setminus Q_1^{\{x_0 \neq 0\}} \cap Q_2.$$

It is defined by the following polynomials:

$$\begin{aligned} q_1(x_0, \dots, x_n)|_{x_0=0} &= h(x_2, \dots, x_n) \\ q_2(x_0, \dots, x_n)|_{x_0=0} &= x_1 L_1(0, x_2, \dots, x_n) + R(x_2, \dots, x_n) \end{aligned}$$

Now we will calculate the class of the intersection of $Q_1 \cap Q_2$ with the complement of U_0 . Consider the projection map

$$\begin{aligned} \pi : \quad \mathbb{P}_k^{n-1} \setminus \{p\} &\longrightarrow \mathbb{P}_k^{n-2} \\ [x_1 : \dots : x_n] &\longmapsto [x_2 : \dots : x_n] \end{aligned}$$

where $p := [1 : 0 : \dots : 0]$. Now this projection map induces an isomorphism

$$(Q_1^{\{x_0=0\}} \cap Q_2) \setminus ((V(L_1) \cap V(x_0)) \cup \{p\}) \cong (V(h) \cap V(x_0)) \setminus ((V(h) \cap V(L_1) \cap V(x_0)))$$

and thus

$$[Q_1^{\{x_0=0\}} \cap Q_2 \setminus ((V(L_1) \cap V(x_0)) \cup \{p\})] = [V(h) \cap V(x_0)] - [V(h) \cap V(L_1) \cap V(x_0)] \quad (7.0.4)$$

in $K_0(Var_k)$.

Since $n - 2 \geq 2$, we have $[V(h) \cap V(x_0)] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(Var_k)$. Besides, the projection map π induces a Zariski locally trivial \mathbb{A}_k^1 -fibration

$$Q_1^{\{x_0=0\}} \cap Q_2 \cap V(L_1) \setminus \{p\} \longrightarrow V(L_1) \cap V(R) \cap V(h) \cap V(x_0).$$

Hence we have

$$[Q_1^{\{x_0=0\}} \cap Q_2 \cap V(L_1) \setminus \{p\}] = \mathbb{L} \cdot [V(L_1) \cap V(R) \cap V(h) \cap V(x_0)] \quad (7.0.5)$$

in $K_0(Var_k)$. By Equality 7.0.4 and Equality 7.0.5, we obtain

$$\begin{aligned} [Q_1^{\{x_0=0\}} \cap Q_2] &= [Q_1^{\{x_0=0\}} \cap Q_2 \setminus ((V(L_1) \cap V(x_0)) \cup \{p\})] + \\ &\quad [Q_1^{\{x_0=0\}} \cap Q_2 \cap V(L_1) \setminus \{p\}] + 1 \\ &= [V(h) \cap V(x_0)] - [V(h) \cap V(L_1) \cap V(x_0)] + \\ &\quad \mathbb{L} \cdot [V(L_1) \cap V(R) \cap V(h) \cap V(x_0)] + 1 \\ &\equiv 2 - [V(h) \cap V(L_1) \cap V(x_0)] \pmod{\mathbb{L}} \end{aligned} \quad (7.0.6)$$

in $K_0(Var_k)$. Here let us note that $V(L_1) \cap V(y_1) = V(L_1) \cap V(x_0)$. Therefore, putting Congruence 7.0.3 and Congruence 7.0.6 together, we get

$$\begin{aligned} [Q_1 \cap Q_2] &= [Q_1^{\{x_0 \neq 0\}} \cap Q_2] + [Q_1^{\{x_0=0\}} \cap Q_2] \\ &= [Y] - [Y \cap V(y_1)] + [Q_1^{\{x_0=0\}} \cap Q_2] \\ &\equiv [Y] \pmod{\mathbb{L}} \end{aligned} \quad (7.0.7)$$

in $K_0(Var_k)$. Now let us examine the class of Y in $K_0(Var_k)$. We consider the following subvariety of Y

$$\begin{aligned} S := \{[y_1 : \dots : y_n] \in \mathbb{P}_k^{n-1} \mid y_1 = h(y_2, \dots, y_n) = L_1(y_1, \dots, y_n) \\ = R(y_2, \dots, y_n) = 0\}. \end{aligned}$$

For each $s := [s_1 : \cdots : s_n] \in S$, we have

$$\frac{\partial \bar{g}}{\partial y_1}(s) = 2s_1 L_0(s) + s_1^2 \frac{\partial L_0}{\partial y_1}(s) + h(s) \frac{\partial L_1}{\partial y_1}(s) = 0,$$

and for $2 \leq i \leq n$,

$$\frac{\partial \bar{g}}{\partial y_i}(s) = s_1^2 \frac{\partial L_0}{\partial y_i}(s) + \frac{\partial h}{\partial y_i}(s) L_1(s) + h(s) \frac{\partial L_1}{\partial y_i}(s) + s_1 \frac{\partial R}{\partial y_i}(s) = 0.$$

Hence $S \subset \text{Sing}(Y)$. Now for $n \geq 5$, $S \neq \emptyset$, therefore $\text{Sing}(Y) \neq \emptyset$, which implies that $[Y] \equiv 1 \pmod{\mathbb{L}}$ in $K_0(\text{Var}_k)$, by Theorem 6.1. In the case that $n = 4$, $Y \subset \mathbb{P}_k^3$ is in general smooth. However, in \mathbb{P}_k^3 a smooth cubic surface is always rational [23, Chapter IV, Theorem 24.1]. Thus

$$[Y] \equiv 1 \pmod{\mathbb{L}} \text{ in } K_0(\text{Var}_k) \quad (7.0.8)$$

for all $n \geq 4$. Hence we have

$$\begin{aligned} [X] &= [Q_1] + [Q_2] - [Q_1 \cap Q_2] \\ &= 2 - [Q_1 \cap Q_2], && \text{by Congruence 7.0.1,} \\ &\equiv 2 - [Y] \pmod{\mathbb{L}}, && \text{by Congruence 7.0.7,} \\ &\equiv 1 \pmod{\mathbb{L}}, && \text{by Congruence 7.0.8} \end{aligned}$$

in $K_0(\text{Var}_k)$.

□

COROLLARY 7.2. *Quartic hypersurfaces of the form described in Theorem 7.1 gives a positive answer to Question 3.7.*

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